Problem 5

What about the PDE

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = 0$$

Can you find all solutions $u(x, y)$ to this equation? (How many are there?) How does this compare with an ODE like

$$\frac{d^2 y}{dx^2} = 0$$

insofar as the number of solutions is concerned?

Solution

$$\frac{\partial}{\partial x}\left[\frac{\partial u(x,y)}{\partial y}\right] = 0$$

Integrate both sides partially with respect to x to undo the partial derivative on the left side.

$$\int^x \frac{\partial}{\partial x'}\left[\frac{\partial u(x',y)}{\partial y}\right] dx' = \int^x 0\, dx'$$

Use the fundamental theorem of calculus on the left. Evaluate the integral on the right.

$$\frac{\partial u(x,y)}{\partial y} = 0 + f(y)$$

Integrate both sides partially with respect to y to undo the partial derivative on the left side.

$$\int^y \frac{\partial u(x,y')}{\partial y'}\, dy' = \int^y f(y')\, dy'$$

Since f is an arbitrary function, its antiderivative is another arbitrary function F.

$$u(x,y) = F(y) + g(x)$$

Here g is another arbitrary function. There are infinitely many solutions for u, for example,

$$u(x,y) = x + y \qquad u(x,y) = \sin x + e^y \qquad u(x,y) = \frac{\cos(\sqrt{x}) + \sin(e^x)}{\sinh x} + \ln y.$$

To solve

$$\frac{d^2 y}{dx^2} = 0,$$

integrate both sides with respect to x once.

$$\frac{dy}{dx} = C_1$$

Integrate both sides with respect to x again.

$$y(x) = C_1 x + C_2$$

Although there are infinitely many solutions for y as well, the functional form (a line) is fixed.

$$y(x) = x + 1 \qquad y(x) = 2x - 5 \qquad y(x) = 16x - 4.13$$